Correspondence

Higher Order Modes in Coupled Helices*

The problem of electromagnetic wave propagation in a system consisting of the two infinitely long concentric sheath helices in free space, shown in Fig. 1, may be approached as a boundary value problem. This has been done before for the case of the lowest mode,[1] but the determinantal equation which gives the propagation constants of the higher order modes has not been given before. This note describes the analysis which gives the propagation constants for all modes of propagation.

The expressions for the field components may be written in terms of the cylindrical wave functions.[2]

Region 1, $(r \leq r_1)$:

$$E_r^{(1)} = \left[\frac{i\beta}{\tau} I_n'(\tau r) a_n^{(1)} - \frac{\mu\omega n}{\tau^2 r} I_n(\tau r) b_n^{(1)}\right] F_n \quad (1a)$$

$$E_\theta^{(1)} = \left[\frac{-n\beta}{\tau^2 r} I_n(\tau r) a_n^{(1)} - \frac{i\mu\omega}{\tau} I_n'(\tau r) b_n^{(1)}\right] F_n \quad (1b)$$

$$E^{(1)} = [-I_n(\tau r) a_n^{(1)}] F_n \quad (1c)$$

$$H_r^{(1)} = \left[\frac{n\omega\epsilon}{\tau^2 r} I_n(\tau r) a_n^{(1)} + \frac{i\beta}{\tau} I_n'(\tau r) b_n^{(1)}\right] F_n \quad (1d)$$

$$H_\theta^{(1)} = \left[\frac{i\omega\epsilon}{\tau} I_n'(\tau r) a_n^{(1)} - \frac{n\beta}{\tau^2 r} I_n(\tau r) b_n^{(1)}\right] F_n \quad (1e)$$

$$H_z^{(1)} = [-I_n(\tau r) b_n^{(1)}] F_n. \quad (1f)$$

Region 2, $(r_1 \leq r \leq r_2)$:

$$E_r^{(2)} = \left[\frac{i\beta}{\tau} I_n'(\tau r) a_n^{(2)} - \frac{\mu\omega n}{\tau^2 r} I_n(\tau r) b_n^{(2)} + \frac{i\beta}{\tau} K_n'(\tau r) c_n^{(2)} - \frac{\mu\omega n}{\tau^2 r} K_n(\tau r) d_n^{(2)}\right] F_n \quad (2a)$$

$$E_\theta^{(2)} = \left[\frac{-n\beta}{\tau^2 r} I_n(\tau r) a_n^{(2)} - \frac{i\mu\omega}{\tau} I_n'(\tau r) b_n^{(2)} - \frac{n\beta}{\tau^2 r} K_n(\tau r) c_n^{(2)} - \frac{i\mu\omega}{\tau} K_n'(\tau r) d_n^{(2)}\right] F_n \quad (2b)$$

$$E_z^{(2)} = [-I_n(\tau r) a_n^{(2)} - K_n(\tau r) c_n^{(2)}] F_n \quad (2c)$$

$$H_r^{(2)} = \left[\frac{n\omega\epsilon}{\tau^2 r} I_n(\tau r) a_n^{(2)} + \frac{i\beta}{\tau} I_n'(\tau r) b_n^{(2)} + \frac{n\omega\epsilon}{\tau^2 r} K_n(\tau r) c_n^{(2)} + \frac{i\beta}{\tau} K_n'(\tau r) d_n^{(2)}\right] F_n \quad (2d)$$

$$H_\theta^{(2)} = \left[\frac{i\omega\epsilon}{\tau} I_n'(\tau r) a_n^{(2)} - \frac{n\beta}{\tau^2 r} I_n(\tau r) b_n^{(2)} + \frac{i\omega\epsilon}{\tau} K_n'(\tau r) c_n^{(2)} - \frac{n\beta}{\tau^2 r} K_n(\tau r) d_n^{(2)}\right] F_n \quad (2e)$$

$$H_z^{(2)} = [-I_n(\tau r) b_n^{(2)} - K_n(\tau r) d_n^{(2)}] F_n. \quad (2f)$$

Region 3, $(r \geq r_2)$:

$$E_r^{(3)} = \left[\frac{i\beta}{\tau} K_n'(\tau r) a_n^{(3)} - \frac{\mu\omega n}{\tau^2 r} K_n(\tau r) b_n^{(3)}\right] F_n \quad (3a)$$

$$E_\theta^{(3)} = \left[\frac{-n\beta}{\tau^2 r} K_n(\tau r) a_n^{(3)} - \frac{i\mu\omega}{\tau} K_n'(\tau r) b_n^{(3)}\right] F_n \quad (3b)$$

$$E_z^{(3)} = [-K_n(\tau r) a_n^{(3)}] F_n \quad (3c)$$

$$H_r^{(3)} = \left[\frac{n\omega\epsilon}{\tau^2 r} K_n(\tau r) a_n^{(3)} + \frac{i\beta}{\tau} K_n'(\tau r) b_n^{(3)}\right] F_n \quad (3d)$$

$$H_\theta^{(3)} = \left[\frac{i\omega\epsilon}{\tau} K_n'(\tau r) a_n^{(3)} - \frac{n\beta}{\tau^2 r} K_n(\tau r) b_n^{(3)}\right] F_n \quad (3e)$$

$$H_z^{(3)} = [-K_n(\tau r) b_n^{(3)}] F_n \quad (3f)$$

where

$$F_n = \exp[in\theta + i\beta z - i\omega t], \quad (4)$$

and

$$\tau = \sqrt{\beta^2 - k^2}.$$

Now in general we should sum over-all values of n and β in (1), (2) and (3); however, since the boundary conditions for sheath helices are the same for all θ and z, the orthogonality in θ and z allows the boundary conditions to be satisfied for each n and β.

We now apply the boundary conditions that $E_\parallel$ (the component of E parallel to the direction of conduction) be zero at all boundaries and that $E_\perp$ (the component of E perpendicular to the direction of conduction) and $H_\parallel$ be continuous across all boundaries. These conditions lead to the following results:

At $r = r_1$:

$$E_z^{(1)} + E_\theta^{(1)} \cot\psi_1 = 0 \quad (5a)$$

$$E_\theta^{(1)} - E_\theta^{(2)} = 0 \quad (5b)$$

$$E_z^{(1)} - E_z^{(2)} = 0 \quad (5c)$$

$$[H_z^{(1)} - H_z^{(2)}] + [H_\theta^{(1)} - H_\theta^{(2)}] \cot\psi_1 = 0, \quad (5d)$$

and likewise at $r = r_2$:

$$E_z^{(3)} + E_\theta^{(3)} \cot\psi_2 = 0 \quad (6a)$$

$$E_\theta^{(2)} - E_\theta^{(3)} = 0 \quad (6b)$$

$$E_z^{(2)} - E_z^{(3)} = 0 \quad (6c)$$

$$[H_z^{(2)} - H_z^{(3)}] + [H_\theta^{(2)} - H_\theta^{(3)}] \cot\psi_2 = 0. \quad (6d)$$

We may now substitute (1), (2), and (3) into (5) and (6) obtaining eight simultaneous equations. These equations may be written

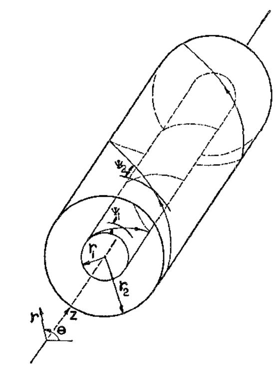

Fig. 1—Concentric sheath helices showing definitions of coordinates and dimensions. The arrows on the sheaths show the direction of conduction.

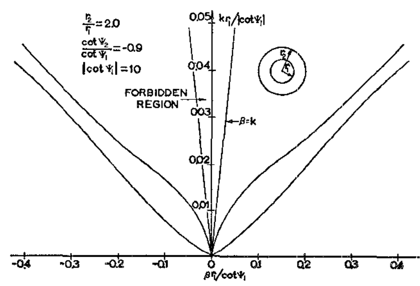

Fig. 2—Natural modes of propagation for $n=0$.

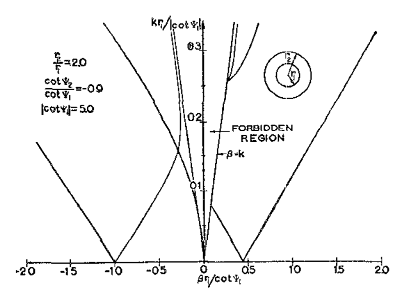

Fig. 3—Natural modes of propagation for $n=+1$. The curves for $n=-1$ are obtained by reversing the signs of the abscissa.

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[1] J. S. Cook, R. Kompfner, and C. F. Quate, "Coupled helices," *Bell Sys. Tech. J.*, vol. 35, pp. 127–178; January, 1956.

[2] J. A. Straton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., p. 360; 1941.

in a more compact form by defining the following notation:

$$A_1 = \frac{n\beta}{\tau^2 r_1} \quad A_2 = \frac{n\beta}{\tau^2 r_2} \qquad B = \frac{i\omega\epsilon}{\tau}$$

$$c = \frac{i\mu\omega}{\tau} \quad I_{n1} = I_n(\tau r_1) \quad K_{n2} = K_n(\tau r_2), \text{etc.}$$

Writing the equation with this notation and removing the common factor, F_n, we have:

$$I_{n1}(1 + A_1 \cot \psi_1)a_n^{(1)} + c \cot \psi_1 I_{n1}' b_n^{(1)} = 0 \quad \text{(7a)}$$

$$A_1 I_{n1} a_n^{(1)} + c I_{n1}' b_n^{(1)} - A_1 I_{n1} a_n^{(2)} - c I_{n1}' b_n^{(2)} - A_1 K_{n1} c_n^{(2)} - c K_{n1}' d_n^{(2)} = 0 \quad \text{(7b)}$$

$$I_{n1} a_n^{(1)} - I_{n1} a_n^{(2)} - K_{n1} c_n^{(2)} = 0 \quad \text{(7c)}$$

$$B \cot \psi_1 I_{n1}' a_n^{(1)} - I_{n1}(1 + A_1 \cot \psi_1) b_n^{(1)} - B \cot \psi_1 I_{n1}' a_n^{(2)} + I_{n1}(1 + A_1 \cot \psi_1) b_n^{(2)} - B \cot \psi_1 K_{n1}' c_n^{(2)} + K_{n1}(1 + A_1 \cot \psi_1) d_n^{(2)} = 0 \quad \text{(7d)}$$

$$K_{n2}(1 + A_2 \cot \psi_2) a_n^{(3)} + c \cot \psi_2 K_{n2}' b_n^{(3)} \quad \text{(7e)}$$

$$A_2 I_{n2} a_n^{(2)} + c I_{n2}' b_n^{(2)} + A_2 K_{n2} c_n^{(2)} + c K_{n2}' d_n^{(2)} - A_2 K_{n2} a_n^{(3)} - c K_{n2}' b_n^{(3)} = 0 \quad \text{(7f)}$$

$$I_{n2} a_n^{(2)} + K_{n2} c_n^{(2)} - K_{n2} a_n^{(3)} = 0 \quad \text{(7g)}$$

$$B \cot \psi_2 I_{n2}' a_n^{(2)} - I_{n2}(1 + A_2 \cot \psi_2) b_n^{(2)} + B \cot \psi_2 K_{n2}' c_n^{(2)} - K_{n2}(1 + A_2 \cot \psi_2) d_n^{(2)} - B \cot \psi_2 K_{n2}' a_n^{(3)} + K_{n2}(1 + A_2 \cot \psi_2) b_n^{(3)} = 0. \quad \text{(7h)}$$

In order to find the permissible values of τ we must set the system determinant equal to zero. The expansion of the eighth order determinant is carried out elsewhere.[3] In order to write the determinantal equation in a compact form we define

$$P_{n1} = I_{n1} K_{n1}, \qquad P_{n1}' = I_{n1}' K_{n1}',$$

$$R_n = I_{n1} K_{n2}, \qquad R_n' = I_{n1}' K_{n2}',$$

$$P_{n2} = I_{n2} K_{n2}, \qquad P_{n2}' = I_{n2}' K_{n2}',$$

$$x = \tau r_1, \qquad y = k r_1 \cot \psi_1,$$

and

$$\xi = n \frac{\sqrt{\cot^2 \psi_1 x^2 + y^2}}{x^2} \cdot \frac{\cot \psi_1}{|\cot \psi_1|}.$$

The general determinantal equation is then found to be

$$[(1 + \xi)^2 P_{n1} + y^2/x^2 P_{n1}'] \cdot \left[\left(1 + \frac{r_1}{r_2} \frac{\cot \psi_2}{\cot \psi_1} \xi\right)^2 P_{n2} + \left(\frac{\cot \psi_2}{\cot \psi_1}\right)^2 y^2/x^2 P_{n2}'\right]$$
$$- \left[\left(1 + \frac{r_1}{r_2} \frac{\cot \psi_2}{\cot \psi_1} \xi\right)^2 R_n + \frac{\cot \psi_2}{\cot \psi_1} y^2/x^2 R_n'\right]$$
$$\cdot \left[(1 + \xi)\left(1 + \frac{r_1}{r_2} \frac{\cot \psi_2}{\cot \psi_1} \xi\right) R_n + y^2/x^2 \frac{\cot \psi_2}{\cot \psi_1} R_n'\right] = 0. \quad (8)$$

[3] R. E. Hayes, "A Study of Coupled Helices," M.S. thesis, University of Kansas, Lawrence; May, 1959.

The values of τ which satisfy (8) give the propagation constants of the various modes of propagation. Setting $n=0$ in (8) we obtain the previously known expression for the lowest mode.[1] This mode is commonly used in the analysis of helical couplers for traveling-wave tubes. Letting r_2 approach infinity we obtain the equation for the modes on a single helix.[4] These special cases indicate the correctness of (8).

The solutions of (8) were found by an approximate method using a digital computer.[3] The modes of propagation for $n=0$, ± 1 are shown in Fig. 2 and Fig. 3. The curves for $n=0$ were calculated from published data,[1] while Fig. 3 is the result of the computer solution. Similar curves may be found for any value of n. The knowledge of these higher order modes should make it possible to obtain more accurate solutions to the coupled helix problem.

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[4] S. Sensiper, "Electromagnetic Wave Propagation in Helical Conductors," Res. Lab. Electronics, Mass. Inst. Tech., Cambridge, Mass., Tech. Rep. No. 194, p. 7; May, 1951.

The Tetrahedral Junction as a Waveguide Switch*

A junction of two rectangular waveguides which are mutually cross-polarized becomes a magnetically controlled reactive switch when properly loaded by a ferrite rod magnetized longitudinally (see Fig. 1). It is a special case of a novel type of structure for which we propose the name *tetrahedral junction*. As a switch, it possesses:

1) very high insertion loss in the reflecting state, ~60 db;
2) loss in the transmitting state which is lower in principle than that attainable in any similar ferrite-waveguide device, <0.1 db;
3) high switching speed—1 μsec is attainable with conventional circuits and convenient currents;
4) large bandwidth, ~10 per cent;
5) little sensitivity to variations in applied field and saturation magnetization; and
6) small phase and small phase-variations with frequency and applied field in the transmitting state.

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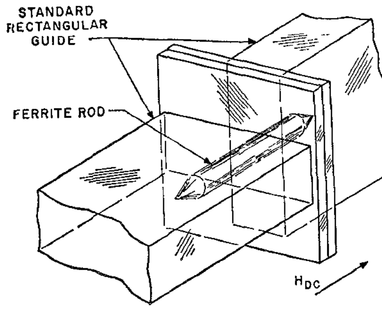

Fig. 1—The tetrahedral junction.

Our experimental and theoretical results indicate that there is room for further improvement in most of its significant properties.

In addition to its utility as a switch, the tetrahedral junction can also be used as a reversible gyrator. Further, its principle of operation is such that a dissipative element can be incorporated which converts the device into a matched modulator or reversible isolator.

The name tetrahedral derives from the fact that if the ends of the two crossed guides are separated and their parallel edges joined by planes, the resulting taper is in the form of a doubly-truncated tetrahedron. Some of our models are of this form, in particular, the one whose properties are reported above.

Our interest in this device is an outgrowth of our study of the Reggia-Spencer phase shifter;[1,2,3] it is one of the "novel effects" to which allusion is made by Weiss.[3] As in the case of the phase shifter, the behavior of the tetrahedral junction may be divided into two regimes: above a sharply defined frequency the device takes on the properties of a Faraday rotator; in a frequency range just below that of the Faraday effect regime, the junction exhibits its most interesting and useful characteristics. Here, however, it is inappropriate to speak of the phase shift regime, for the phase shift effects which are central to the Reggia-Spencer device are inessential, in fact undesirable, in this case. On the other hand, the modes of propagation[3] are of the same basic form in the two. In the presence of the magnetized ferrite, a wave entering the junction of Fig. 1 from the input end takes on a characteristic elliptic polarization. At the plane of the joint it is scattered into four significant components: a reflected and a transmitted propagating mode, and an evanescent mode in each guide which is also elliptically polarized. A model of this phenomenon, employing simplifying assumptions similar to those used by Weiss,[3] shows that under the proper conditions (involving the cross-sectional dimensions of the two guides, and the

[1] F. Reggia and E. G. Spencer, "A new technique in ferrite phase-shifting for beam scanning of microwave antennas," PROC. IRE, vol. 45, pp. 1510–1517; November, 1957.

[2] J. A. Weiss, "The Reggia-Spencer microwave phase shifter," *J. Appl. Phys.*, vol. 30, pp. 153s–154s; April, 1959. *Proc. AIEE Conf. on Magnetism and Magnetic Materials*, Philadelphia, Pa., November, 1958.

[3] J. A. Weiss, "A phenomenological theory of the Reggia-Spencer phase shifter," PROC. IRE, vol. 47, pp. 1130–1137; June, 1959.